

CONVERGENCE OF OPERATOR SEMIGROUPS ASSOCIATED WITH GENERALISED ELLIPTIC FORMS

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ABSTRACT. In a recent article, Arendt and ter Elst have shown that every sectorial form is in a natural way associated with the generator of an analytic strongly continuous semigroup, even if the form fails to be closable. As an intermediate step they have introduced so-called *j-elliptic* forms, which generalises the concept of elliptic forms in the sense of Lions. We push their analysis forward in that we discuss some perturbation and convergence results for semigroups associated with *j-elliptic* forms. In particular, we study convergence with respect to the trace norm or other Schatten norms. We apply our results to Laplace operators and Dirichlet-to-Neumann-type operators.

1. INTRODUCTION

The use of sesquilinear form in semigroup theory dates back to the works of Tosio Kato and Jacques-Louis Lions. A generalisation of Kato's and Lions' approach has been recently proposed by Wolfgang Arendt and Tom ter Elst [3]. Their method permits to treat differential operators on rough domains, strongly degenerate equations, Dirichlet-to-Neumann operators and Stokes-type equations with ease, cf. [3, 5]. In this article we consider only what they call the *complete case*, which corresponds to Lions' forms, not their *incomplete case*, which corresponds to Kato's approach. These two notions are different descriptions of the same ideas. In Section 2 we introduce *j-elliptic* forms and recall some basic facts which we need. We also prove that *j-ellipticity* is preserved under small perturbations and we also present a generalisation of the Courant's minimax formula.

The study of convergence of sequences of C_0 -semigroups goes back to the pioneering works on semigroup theory in the 1950s. In particular, convenient convergence criteria for semigroups associated with closed forms can be found in Kato's book [27]. In Section 3 we establish criteria for *j-elliptic* forms that imply strong convergence of the associated semigroups. Such convergence results will in turn allow us to deduce convergence in stronger norms, for example Schatten norms. Our first result in this section is a Mosco-like convergence criterion for symmetric forms (Theorem 3.1).

The Schatten classes \mathcal{L}_p have been introduced in [39] by Robert Schatten and John von Neumann. For $p = 1$, one obtains the well-studied trace class. It became clear soon after the publication of [39] that trace class operators play an important rôle in spectral theory, perturbation theory and mathematical physics. An interesting account on the history of the development of the Schatten theory can be found in the introduction of [40]. Criteria for convergence of a sequence of operators with respect to Schatten norms have been investigated for a long time, see e.g. [44] and references therein. We translate a result due to Valentin A. Zagrebnov into the framework of *j-elliptic* forms, which gives a sufficient condition for convergence in Schatten norm. We then combine this with an interpolation result for Schatten class

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operators in order to prove convergence of semigroups as Schatten class operators into spaces of higher regularity, e.g. from $L^2(\Omega)$ into $H^k(\Omega)$ for any $k \in \mathbb{N}$.

Summarizing our main results, on $L^2(X)$, X a finite measure space, the following holds:

Strong convergence implies trace norm (hence uniform) convergence of a family of self-adjoint contraction semigroups, provided that their generators all dominate the generator of an ultra-contractive semigroup; and this even as operators from $L^2(X)$ into a space of more regular functions.

This is made precise in Corollary 3.8 and the subsequent remark.

In Section 4 we present several applications for our theorems and ideas. More precisely, we study Schatten norm convergence of semigroups generated by Laplacians with varying Robin boundary conditions as well as trace norm convergence of semigroups generated by Dirichlet-to-Neumann-like operators with varying coefficients. We also compare the spectra of several self-adjoint operators based on our general version of the minimax formula.

2. GENERALISED ELLIPTIC FORMS

In this section we study *j-elliptic forms*. We start with some basic facts. For a broader introduction and proofs of the fundamental theorems we refer to [3].

Definition 2.1. *Let V and H be Hilbert spaces and $j: V \rightarrow H$ a bounded linear map with dense range. A sesquilinear form $a: V \times V \rightarrow \mathbb{C}$ is called a *j-elliptic form* on H with form domain V if it is continuous as a function from $V \times V$ to \mathbb{C} and there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that*

$$(2.1) \quad \operatorname{Re} a(u, u) - \omega \|j(u)\|_H^2 \geq \mu \|u\|_V^2 \quad \text{for all } u \in V.$$

The unique, densely defined, m -sectorial operator A on H given by

$$\begin{aligned} D(A) &:= \{x \in H : \exists u \in V, j(u) = x, \exists f \in H \text{ s.t. } a(u, v) = (f | j(v))_H \ \forall v \in V\} \\ Ax &:= f \end{aligned}$$

is called the operator associated with (a, j) . We say that (a, j) is associated to A and also that (a, j) is associated with the analytic C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on H .

We say that a is symmetric if $a(u, v) = \overline{a(v, u)}$ for all $u, v \in V$. In this case the associated operator A and the semigroup $(e^{-tA})_{t \geq 0}$ are self-adjoint.

We say that a is positive if $a(u, u) \geq 0$ for all $u \in V$. By the polarisation identity every positive (or, more generally, every real-valued) sesquilinear form is symmetric.

If j is injective, we can regard V as a subspace of H , regarding j as the embedding of V into H . In this case the notion of a *j-elliptic form* a introduced in Definition 2.1 coincides with Lions' definition of elliptic forms. Thus we refer to this situation as *classical*, i.e., we say that (a, j) is a *classical form* if j is injective.

Remark 2.2. Let a be a *j-elliptic form*. Then

$$(2.2) \quad V(a) := \{u \in V : a(u, v) = 0 \ \forall v \in \ker j\}$$

is a closed subspace of V , $j|_{V(a)}$ is injective and $V = V(a) \oplus \ker j$. In particular, $j(V(a)) = j(V)$ is a dense subspace of H and $j|_{V(a)}$ is injective. The classical form $(a|_{V(a) \times V(a)}, j|_{V(a)})$ is associated to the same operator as (a, j) . This relation allows us to carry over many results about classical forms to *j-elliptic forms*, which is the basis of this section.

Remark 2.3. Remark 2.2 suggests that a is associated with an m -sectorial operator if merely

$$\operatorname{Re} a(u, u) - \omega \|j(u)\|_H^2 \geq \mu \|u\|_V^2 \quad \text{for all } u \in V(a).$$

This is indeed true provided that we require $V = V(a) + \ker j$ in addition, cf. [3, Cor. 2.2].

We want to extend several classical results to j -elliptic forms. We begin with a generation result which is a translation of a celebrated by Michel Crouzeix on cosine function generators.

Proposition 2.4. *Let a be a j -elliptic form and denote by A the associated operator. Assume that there exists $M \geq 0$ such that*

$$(2.3) \quad |\operatorname{Im} a(u, u)| \leq M \|u\|_V \|j(u)\|_H \quad \text{for all } u \in V(a).$$

Then $-A$ generates a cosine operator function and hence a semigroup with analyticity angle of $\frac{\pi}{2}$.

Proof. For all $x \in D(A)$ with $\|x\|_H = 1$ there exists $u \in V(a)$ such that $j(u) = x$ and

$$\begin{aligned} |\operatorname{Im}(Ax|x)_H|^2 &= |\operatorname{Im} a(u, u)|^2 \leq M^2 \|u\|_V^2 \|j(u)\|_H^2 \\ &\leq \frac{M^2}{\mu} (\operatorname{Re} a(u, u) - \omega \|j(u)\|_H^2) \|j(u)\|_H^2 \\ &= \frac{M^2}{\mu} (\operatorname{Re}(Ax|x)_H - \omega). \end{aligned}$$

Thus, the numerical range of A is contained in a parabola and therefore $-A$ generates a cosine operator function by Crouzeix' celebrated result [17]. Finally, every generator of a cosine function family generates a holomorphic semigroup of angle $\frac{\pi}{2}$ [6, Thm. 3.14.17]. \square

The following perturbation results are analogous to two classical perturbation theorems for operators [19, 20], one relying on interpolation estimates, the other one on compactness.

Proposition 2.5. *Let $a : V \times V \rightarrow \mathbb{C}$ be a j -elliptic form and let H' be a subspace of H containing $j(V)$. Let H' carry its own norm $\|\cdot\|_{H'}$, for which it is a Banach space and is continuously embedded into H . Assume that there exist $\alpha \in [0, 1)$ and $M \geq 0$ such that*

$$\|j(u)\|_{H'} \leq M \|u\|_V^\alpha \|j(u)\|_H^{1-\alpha} \quad \text{for all } u \in V.$$

Let $b : V \times V \rightarrow \mathbb{C}$ be a continuous sesquilinear form such that

$$\operatorname{Re} b(u, u) \geq -c \|u\|_V \|j(u)\|_{H'} \quad \text{for all } u \in V$$

for some $c \geq 0$. Then $a + b : V \times V \rightarrow \mathbb{C}$ is j -elliptic.

Proof. We apply Young's inequality $\alpha\beta \leq \varepsilon\alpha^p + c_{\varepsilon,p}\beta^{p/(p-1)}$, which is valid for every $p \in (1, \infty)$, every $\alpha, \beta \geq 0$, and every $\varepsilon > 0$ with some constant $c_{\varepsilon,p} \geq 0$. For $p := \frac{2}{1+\alpha}$ we obtain that

$$\begin{aligned} \operatorname{Re} b(u, u) &\geq -c \|u\|_V \|j(u)\|_{H'} \geq -cM \|u\|_V^{1+\alpha} \|j(u)\|_H^{1-\alpha} \\ &\geq -cM\varepsilon \|u\|_V^2 - cMc_{\varepsilon,p} \|j(u)\|_H^2 \end{aligned}$$

for all $u \in V$. For $\varepsilon := \frac{\mu}{2cM}$ we thus obtain that

$$\operatorname{Re} a(u, u) + \operatorname{Re} b(u, u) - (\omega - cMc_{\varepsilon,p}) \|j(u)\|_H^2 \geq \frac{\mu}{2} \|u\|_V^2$$

for all $u \in V$, which is the claim. \square

Remark 2.6. In the classical case Proposition 2.5 coincides with [33, Lemma 2.1].

For the second perturbation theorem we need the following simple lemma.

Lemma 2.7. *Let V be a reflexive Banach space, $T: V \rightarrow H$ an injective bounded linear operator into a Banach space H and $S: V \rightarrow Z$ a compact linear operator into a Banach space Z . Then for every $\varepsilon > 0$ there exists $c_\varepsilon \geq 0$ such that*

$$\|Su\|_Z \leq \varepsilon\|u\|_V + c_\varepsilon\|Tu\|_H \quad \text{for all } u \in V.$$

Proof. Assume to the contrary that there exist $\varepsilon_0 > 0$ and a sequence $(u_n)_{n \in \mathbb{N}} \subset V$ such that

$$\|Su_n\|_Z \geq \varepsilon_0\|u_n\|_V + n\|Tu_n\|_H \quad \text{for all } n \in \mathbb{N}.$$

We can assume that $\|Su_n\|_Z = 1$ after rescaling. Passing to a subsequence we have $u_n \rightharpoonup u$ in V , hence $Tu_n \rightharpoonup Tu$ in H . Now $\|Tu_n\|_H \leq \frac{1}{n}$ implies that $Tu = 0$ and thus $u = 0$. Hence by compactness $\lim_{n \rightarrow \infty} Su_n = Su = 0$ in Z , contradicting $\|Su_n\|_Z = 1$. \square

The conclusion of the following perturbation result should be compared with Remark 2.3.

Proposition 2.8. *Let a be a j -elliptic form on V . Let S be a compact operator from V into a Banach space Z and let $b_0: V \times Z \rightarrow \mathbb{C}$ be a bounded sesquilinear form. Define $b(u, v) := b_0(u, Sv)$ on $V \times V$. If j is injective on $V(a+b)$, where $V(a+b)$ is defined as in (2.2), then there exist $\omega' \in \mathbb{R}$ and $\mu' > 0$ such that*

$$(2.4) \quad \operatorname{Re} a(u, u) + \operatorname{Re} b(u, u) - \omega'\|j(u)\|_H^2 \geq \mu'\|u\|_V^2 \quad \text{for all } u \in V(a+b).$$

Proof. Regarding j as an injective operator on $V(a+b)$, from Lemma 2.7 we obtain that

$$\|Su\|_Z \leq \varepsilon\|u\|_V + c_\varepsilon\|j(u)\|_H$$

for all $u \in V(a+b)$. Hence

$$\begin{aligned} |b(u, u)| &= |b_0(u, Su)| \leq c\|u\|_V\|Su\|_Z \\ &\leq \varepsilon c\|u\|_V^2 + c_\varepsilon c\|u\|_V\|j(u)\|_H \leq \varepsilon c\|u\|_V^2 + \delta c_\varepsilon c\|u\|_V^2 + \frac{c_\varepsilon c}{4\delta}\|j(u)\|_H^2 \end{aligned}$$

for $u \in V(a+b)$ by Young's inequality. If we first pick $\varepsilon > 0$ small enough and then $\delta > 0$, we easily obtain the claimed estimate from the j -ellipticity of a . \square

Strictly speaking, the preceding result is not quite a perturbation result because we leave the class of j -elliptic forms. It is, however, quite useful in situations where one cannot expect that a lower order perturbation preserves j -ellipticity, see [3, §4.4] for such an example.

We continue our investigation of j -elliptic forms with results about domination and convergence. It is well-known that domination of self-adjoint operators in terms of their resolvents can be expressed via their quadratic forms. One implication of this characterisation remains true for symmetric j -elliptic forms. The following proposition is a direct consequence of Remark 2.2 and [27, Thm. VI.2.21].

Proposition 2.9. *Let H be a Hilbert space, let a_1 be a symmetric j_1 -elliptic form and let a_2 be a symmetric j_2 -elliptic form, where $j_1: V_1 \rightarrow H$ and $j_2: V_2 \rightarrow H$. Let A_i be the self-adjoint operator on H which is associated with a_i , $i = 1, 2$. We say that a_1 lies above a_2 (and write $(a_1, j_1) \geq (a_2, j_2)$) if*

- (1) $j_1(V_1) \subset j_2(V_2)$ and
- (2) $a_1(u_1, u_1) \geq a_2(u_2, u_2)$ whenever $j_1(u_1) = j_2(u_2)$.

In this case $(\gamma + A_1)^{-1} \leq (\gamma + A_2)^{-1}$ in the sense of positive definite operators for all sufficiently large $\gamma \in \mathbb{R}$.

We also give a result concerning the domination of the spectra in the case where reference spaces H differ. The following is an easy consequence of the Courant–Fischer theorem for self-adjoint operators (or, rather, their quadratic forms) and Remark 2.2.

Lemma 2.10. *Let a be a symmetric j -elliptic form on a Hilbert space H with form domain V and associated operator A . If j is compact, then the self-adjoint operator A has compact resolvent, and we can order the eigenvalues of A in increasing order, i.e.,*

$$\lambda_1(A) \leq \lambda_2(A) \leq \lambda_3(A) \leq \cdots \leq \lambda_n(A) \rightarrow \infty,$$

taking into account multiplicities. In this case, the eigenvalues are given by the min-max principle

$$\lambda_k(A) = \min_{\substack{E \subset V(a) \\ \dim E = k}} \max_{\substack{u \in E \\ u \neq 0}} \frac{a(u, u)}{\|j(u)\|_H^2},$$

i.e., E runs over the k -dimensional subspaces of $V(a)$.

The following theorem allows the comparison of operators on different spaces that have comparable j -elliptic forms.

Theorem 2.11. *Let V_1, V_2, H_1 and H_2 be Hilbert spaces such that V_2 is a closed subspace of V_1 , which is equipped with the norm of V_1 . Let a_1 be a symmetric j_1 -elliptic form, where $j_1: V_1 \rightarrow H_1$ is compact, and let a_2 be a symmetric j_2 -elliptic form, where $j_2: V_2 \rightarrow H_2$ is bounded. Assume that $\ker j_1 \subset V_2$ and that*

$$(2.5) \quad \|j_1(u)\|_{H_1} \geq \|j_2(u)\|_{H_2} \quad \text{and} \quad a_1(u, u) \leq a_2(u, u) \quad \text{for all } u \in V_2.$$

Then j_2 is compact and $\lambda_k(A_1) \leq \lambda_k(A_2)$ for all $k \in \mathbb{N}$, where A_1 and A_2 are the operators associated with a_1 and a_2 on H_1 and H_2 , respectively.

Proof. Let (u_n) be a bounded sequence in V_2 . Then (u_n) is a bounded sequence in V_1 . Passing to a subsequence we can assume that $(j_1(u_n))$ converges in H_1 . Since

$$\|j_2(u_n) - j_2(u_m)\|_{H_2} \leq \|j_1(u_n) - j_1(u_m)\|_{H_1}$$

by (2.5) this implies that $(j_2(u_n))$ is a Cauchy sequence in H_2 , hence convergent. We have proved compactness of j_2 .

For the spectral domination it suffices to consider the following three special cases:

- (i) $a_2 = a_1|_{V_2 \times V_2}$ and $j_2 = j_1|_{V_2}$; or
- (ii) $V_1 = V_2$ and $j_1 = j_2$; or
- (iii) $V_1 = V_2$ and $a_1 = a_2$.

In fact, once we have established the result in these situations, we obtain that

$$\lambda_k(a_1, j_1) \leq \lambda_k(a_1|_{V_2 \times V_2}, j_1|_{V_2}) \leq \lambda_k(a_2, j_1|_{V_2}) \leq \lambda_k(a_2, j_2) \quad (k \in \mathbb{N}).$$

Here we have defined $\lambda_k(a, j) := \lambda_k(A)$ with A associated to (a, j) to keep the notation simple. It should be noted that $a_1|_{V_2 \times V_2}$ is $j_1|_{V_2}$ -elliptic since $V_2 \subset V_1$ and a_2 is $j_1|_{V_2}$ -elliptic since $a_2(u, u) \geq a_1(u, u)$ on V_2 .

So let us prove the theorem in those three cases.

- (i) Assume that $a_2 = a_1|_{V_2 \times V_2}$ and $j_2 = j_1|_{V_2}$. Since $\ker j_1 \subset V_2$, this implies that $\ker j_1 = \ker j_2$. Thus trivially $V(a_2) \subset V(a_1)$, see (2.2), implying that every subspace of $V(a_2)$ is a subspace of $V(a_1)$. Hence $\lambda_k(A_1) \leq \lambda_k(A_2)$ for all $k \in \mathbb{N}$ by Lemma 2.10.
- (ii) Assume that $V_1 = V_2 =: V$ and $j_1 = j_2 =: j$. Let $k \in \mathbb{N}$ be arbitrary and fix a subspace E_2 of $V(a_2)$ with $\dim E_2 = k$ such that

$$\lambda_k(A_2) = \max_{\substack{u \in E_2 \\ u \neq 0}} \frac{a_2(u, u)}{\|j(u)\|^2}$$

Then in particular

$$(2.6) \quad \lambda_k(A_2) \geq \max_{\substack{u \in E_2 \\ u \neq 0}} \frac{a_1(u, u)}{\|j(u)\|^2}$$

by (2.5). Define

$$E_1 := \{u \in V(a_1) : j(u) \in j(E_2)\}.$$

Since j is bijective from $V(a_1)$ and $V(a_2)$ to $j(V)$, respectively, see Remark 2.2, we have $\dim E_1 = k$, thus

$$(2.7) \quad \lambda_k(A_1) \leq \max_{\substack{u \in E_1 \\ u \neq 0}} \frac{a_1(u, u)}{\|j(u)\|^2}$$

by Lemma 2.10. In view of (2.6) and (2.7) the theorem is proved once we show that for every $u \in E_1$ there exists $\tilde{u} \in E_2$ such that $a_1(u, u) \leq a_1(\tilde{u}, \tilde{u})$ and $j(u) = j(\tilde{u})$.

Thus fix $u \in E_1 \subset V(a_1)$. By definition of E_1 there exists $\tilde{u} \in E_2$ such that $j(u) = j(\tilde{u})$. By Remark 2.2 there exist $\tilde{u}_1 \in V(a_1)$ and $\tilde{u}_2 \in \ker j$ such that $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$, so in particular $j(u) = j(\tilde{u}) = j(\tilde{u}_1)$. Since j is injective on $V(a_1)$, this implies that $u = \tilde{u}_1$. Hence

$$\begin{aligned} a_1(\tilde{u}, \tilde{u}) &= a_1(u + \tilde{u}_2, u + \tilde{u}_2) \\ &= a_1(u, u) + 2 \operatorname{Re} a_1(u, \tilde{u}_2) + a_1(\tilde{u}_2, \tilde{u}_2) \geq a_1(u, u) \end{aligned}$$

since $a_1(u, \tilde{u}_2) = 0$ by definition of $V(a_1)$ and $a_1(\tilde{u}_2, \tilde{u}_2) \geq 0$ by (2.1).

- (iii) Assume that $V_1 = V_2$ and $a_1 = a_2 =: a$. From (2.5) we obtain that $\ker j_1 \subset \ker j_2$, which implies $V(a_2) \subset V(a_1)$. Now we can proceed as in the first case. \square

For semigroups on $L^2(\Omega)$ associated with classical forms, ultra-contractivity is well-known to be equivalent to an embedding of the form domain into $L^q(\Omega)$ for $q > 2$, provided that the semigroup extends to a contractive semigroup on $L^\infty(\Omega)$. We translate this result into the language of j -elliptic forms, which will be useful in the subsequent sections when we study Gibbs semigroups.

Proposition 2.12. *Let Ω be a σ -finite measure space. Let a be a j -elliptic form on $H := L^2(\Omega)$ with form domain V and associated operator A . Assume that there exists $M \geq 0$ such that $\|e^{-tA}f\|_\infty \leq M\|f\|_\infty$ for all $f \in L^\infty(\Omega) \cap L^2(\Omega)$ and all $t \in [0, 1]$. Assume moreover that $j(V) \subset L^{\frac{2d}{d-2}}(\Omega)$ for some $d > 2$. Then $(e^{ta})_{t \geq 0}$ is ultra-contractive, i.e., $e^{-tA}L^2(\Omega) \subset L^\infty(\Omega)$ and*

$$\|e^{-tA}\|_{\mathcal{L}(L^2, L^\infty)} \leq ct^{-\frac{d}{4}}, \quad t \in (0, 1],$$

for some constant $c > 0$.

Proof. By the closed graph theorem j is bounded from V to $L^{\frac{2d}{d-2}}(\Omega)$. Thus the result follows from Remark 2.2 and [36, Thm. 6.4]. \square

3. CONVERGENCE RESULTS

Several results in [3] are based on a convergence result [3, Thm. 3.9]. We extend this criterion in the case of symmetric forms. It is well-known that for symmetric classical forms the convergence in the sense of Mosco, see [32], is equivalent to strong convergence of the resolvents. In fact, this holds even in the nonlinear case and is typically stated only in that situation. We show how this criterion translates to j -elliptic forms.

Theorem 3.1. *Let $(a_n, j_n)_{n \in \mathbb{N}}$ and (a, j) be positive forms on a Hilbert space H with form domains $(V_n)_{n \in \mathbb{N}}$ and V , respectively. We assume that a_n is j_n -elliptic for all $n \in \mathbb{N}$ and a is j -elliptic. Then the following are equivalent.*

- (a) *The sequence of operators $(-A_n)_{n \in \mathbb{N}}$ associated with $(a_n, j_n)_{n \in \mathbb{N}}$ converges to the operator $-A$ associated with (a, j) in the strong resolvent sense.*
- (b) *The following conditions are satisfied:*
 - (i) *If $u_n \in V_n$, $j_n(u_n) \rightarrow x$ for some $x \in H$ and $\liminf_{n \rightarrow \infty} a_n(u_n, u_n) < \infty$, then there exists $u \in V$ such that $j(u) = x$ and $\liminf_{n \rightarrow \infty} a_n(u_n, u_n) \geq a(u, u)$;*
 - (ii) *For all $u \in V$ there exists a sequence $(u_n)_{n \in \mathbb{N}}$ with $u_n \in V_n$ such that*

$$\lim_{n \rightarrow \infty} j_n(u_n) = j(u) \quad \text{and} \quad \liminf_{n \rightarrow \infty} a_n(u_n, u_n) \leq a(u, u).$$

If these equivalent conditions are satisfied, we say that $(a_n, j_n)_{n \in \mathbb{N}}$ converges to (a, j) in the sense of Mosco.

Proof. Define $\phi_n(j_n(u)) := a_n(u, u)$ for $u \in V_n(a_n)$, and $\phi_n(x) := \infty$ for $x \in H \setminus j_n(V_n)$. Then $\phi_n: H \rightarrow (-\infty, \infty]$ is well-defined, convex and lower semicontinuous, and $-A_n$ is the subdifferential of ϕ_n . This follows from [3, Thm. 2.5] and the well-known correspondence between the linear and the non-linear theory of forms. Moreover,

$$(3.1) \quad \phi_n(x) = \min\{a(u, u) : u \in V_n, j_n(u) = x\}$$

by Remark 2.2. A similar statement holds for the functional ϕ , which we define analogously for (a, j) .

The two conditions in (b) are equivalent to

- (I) $x_n \rightarrow x$ implies that $\liminf_{n \rightarrow \infty} \phi_n(x_n) \geq \phi(x)$;
- (II) for all $x \in H$ there exists $(x_n) \subset H$ such that

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} \phi_n(x_n) = \phi(x).$$

In fact, assume (i) and (ii). If $\liminf_{n \rightarrow \infty} \phi_n(x_n) = \infty$ in (I), then there is nothing to show. Otherwise, (I) follows from (i) and (3.1). In (II), if $x \notin j(V)$, i.e., $\phi(x) = \infty$, then by (I) any sequence (x_n) in H such that $\lim_{n \rightarrow \infty} x_n = x$ does the job. On the other hand, if $x = j(u)$ for some $u \in V(a)$, then (II) follows from (ii) and (I). On the contrary, if (I) and (II) are satisfied, then (i) and (ii) follow easily using (3.1).

We have shown that condition (b) is equivalent to Mosco-convergence of ϕ_n to ϕ , which by [8, Prop. 3.19 and Thm. 3.26] is equivalent to strong resolvent convergence of the subdifferentials, i.e., to (a). \square

Remark 3.2. The implication from (b) to (a) in Theorem 3.1 remains valid for symmetric, but not necessarily positive forms provided that there exists $\omega \leq 0$ such that $a_n(u, u) - \omega \|j_n(u)\|_H^2 \geq 0$ for all $u \in V_n$ and $a(u, u) - \omega \|j(u)\|_H^2 \geq 0$ for all $u \in V$. In fact, assume that the conditions in (b) are fulfilled. Lower semicontinuity of the norm in H yields that then also the positive forms \tilde{a}_n and \tilde{a} given by $\tilde{a}_n(u, v) := a_n(u, v) - \omega (j_n(u) | j_n(v))_H$ and $\tilde{a}(u, v) := a(u, v) - \omega (j(u) | j(v))_H$ satisfy the conditions in (b). Now the theorem implies that the associated operators $(-A_n - \omega)$ converge to $(-A - \omega)$ in the strong resolvent sense, which trivially implies (a).

Let H_1 and H_2 be separable Hilbert spaces. For $p \in [1, \infty)$ the p -Schatten class is defined by

$$\mathcal{L}_p(H_1, H_2) := \{T \in \mathcal{K}(H_1, H_2) : \|T\|_{\mathcal{L}_p} := \|(s_n)_{n \in \mathbb{N}}\|_{\ell^p} < \infty\},$$

where $(s_n)_{n \in \mathbb{N}}$ is the sequence of *singular values* of T , i.e., the sequence of eigenvalues of $|T| := (T^*T)^{\frac{1}{2}}$. Then $\|\cdot\|_{\mathcal{L}_p}$ is a complete norm on $\mathcal{L}_p(H_1, H_2)$, called the

p -Schatten norm. The operators in $\mathcal{L}_1(H_1, H_2)$ are also called *trace class operators* with the *trace norm*, and the operators in $\mathcal{L}_2(H_1, H_2)$ are called *Hilbert–Schmidt operators*. If $H_1 = H_2 = H$ we frequently write $\mathcal{L}_p(H)$ instead of $\mathcal{L}_p(H, H)$. For more information about the Schatten classes we refer to [24, 40].

We are mainly interested in semigroups consisting of Schatten class operators. The following definition goes back to Dietrich A. Uhlenbrock [43] and first appeared in applications in statistical mechanics. Nowadays, Gibbs semigroups are popular objects in mathematical physics.

Definition 3.3. *Let H be a Hilbert space. A Gibbs semigroups is a C_0 -semigroup $(T(t))_{t \geq 0}$ on H such that each operator $T(t)$, $t > 0$, is of trace class.*

Remarks 3.4. (1) Since $\mathcal{L}_p(H) \cdot \mathcal{L}_q(H) \subset \mathcal{L}_r(H) \subset \mathcal{L}_{r'}(H)$ for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $r < r'$, every semigroup $(T(t))_{t \geq 0}$ for which there exists $p \in [1, \infty)$ such that $T(t) \in \mathcal{L}_p(H)$ for all $t > 0$ is a Gibbs semigroup.

(2) Let X be a finite measure space. It is known that each bounded linear operator T from $L^2(X)$ to $L^\infty(X)$ is a Hilbert–Schmidt operator [2, Thm. 1.6.2]. In particular every ultra-contractive semigroup on $L^2(X)$ is a Gibbs semigroup. Hence Proposition 2.12 provides a sufficient condition for the Gibbs property, which is sometimes easy to check.

(3) It seems to be difficult to characterise the Gibbs property in terms of the resolvent. If $-A$ generates an analytic semigroup $(T(t))_{t \geq 0}$ on H and $(\lambda + A)^{-k} \in \mathcal{L}_p(H)$ for some $k \in \mathbb{N}$, some λ in the resolvent set and some $p \in [1, \infty)$, then $(T(t))_{t \geq 0}$ is a Gibbs semigroup. In fact, since in that case the embedding $D(A^k) \hookrightarrow H$ is of Schatten class and $T(t): H \rightarrow D(A^k)$ is bounded for $t > 0$, the ideal property implies that $T(t) \in \mathcal{L}_p(H)$ for all $t > 0$.

But the converse fails. In fact, consider the diagonal operator $A = D_\lambda$ on ℓ^2 and $(T(t))_{t \geq 0} = (e^{-tA})_{t \geq 0}$, where $\lambda_n := \log^2 n$. Then the eigenvalues $e^{-t \log^2 n} = n^{-t \log n}$ of $T(t)$ are summable for every $t > 0$, i.e., $(T(t))_{t \geq 0}$ is a Gibbs semigroup, but the eigenvalues $(\lambda + \log^2 n)^{-k}$ of $(\lambda + A)^{-k}$ are not p -summable for any $k \in \mathbb{N}$, $p \in [1, \infty)$ and λ in the resolvent set.

(4) The square root of the above operator D_λ yields also another interesting counterexample. It is known that for an analytic semigroup immediate compactness and eventual compactness are equivalent. However, the square root of D_λ generates a semigroup whose eigenvalues $e^{-t \log n} = n^{-t}$ are p -summable if and only if $t > 1/p$. In particular, this self-adjoint semigroup is eventually Gibbs, but not immediately Gibbs.

(5) It is known that for a bounded domain $\Omega \subset \mathbb{R}^d$ with the cone property the embedding of $H^k(\Omega)$ into $L^2(\Omega)$ is a Hilbert–Schmidt operator whenever $2k > d$, see [30], and in fact a p -Schatten class operator if $pk > d$, see [25]. Under certain assumptions on the geometry, Maurin’s and Gramsch’s result have been extended to unbounded domains [15, 28]. In such situations, if A generates an analytic semigroup and $D(A) \subset H^1(\Omega)$, then A generates a Gibbs semigroup. Observe that by [1, Thm. 6.54 and Rem. 6.55] there exist domains with infinite measure such that the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is in $\mathcal{L}_p(H^1(\Omega), L^2(\Omega))$ for some $p \in [1, \infty)$. In this situation criterion (2) does not apply.

(6) The preceding criterion can also be useful for semigroups on Sobolev spaces H^s with index $s \neq 0$. For example, it allows us to prove that the semigroup generated by the Wentzell–Robin Laplacian on a smooth domain (see Section 4.7 for details) on $H^1(\Omega)$ considered in [7, §2.9] and [21] is Gibbs. To be more precise, recall that the domain of the Wentzell–Robin–Laplacian is a subspace of $H^{\frac{3}{2}}(\Omega) \times L^2(\partial\Omega)$. Thus, the semigroup generated by its part in $V := \{(u, u|_{\partial\Omega}) : u \in H^1(\Omega)\}$ maps V to $\{(u, u|_{\partial\Omega}) : u \in H^{\frac{3}{2}}(\Omega)\}$ for all $t > 0$.

By [25, Satz 1] the embedding $H^{\frac{3}{2}}(\Omega) \hookrightarrow H^1(\Omega)$ is a p -Schatten class operator for all $p > 2d$, hence so is any operator of the semigroup for $t > 0$, and by part (1) this semigroup is Gibbs. The same argument applies to general (also non-selfadjoint) elliptic operators with Wentzell-Robin or similar boundary conditions. On the other hand, part (2) does not yield the result in this case since the semigroup is not defined on an L^2 -space.

We apply known result about convergence in Schatten norms, cf. [40, Chapter 2], to semigroups arising from j -elliptic forms. The following proposition is a direct consequence of Proposition 2.9 together with [44, Lemma, p.271]. Its conditions are often easy to check; we will give some examples later on.

Theorem 3.5. *Let H be a Hilbert space and let (a_n, j_n) , (a, j) and (b, j) be symmetric, sesquilinear forms that satisfy the conditions in Definition 2.1. We denote by A_n , A and B the associated self-adjoint operators. Assume that*

- (i) $(b, j) \leq (a_n, j_n)$ for all $n \in \mathbb{N}$ in the sense of Proposition 2.9,
- (ii) $-B$ generates a Gibbs semigroup, and
- (iii) (A_n) converges to A in the strong resolvent sense.

Then

$$\lim_{n \rightarrow \infty} e^{-tA_n} = e^{-tA} \quad \text{in } \mathcal{L}_1(H)$$

for every $t > 0$.

Remark 3.6. Theorem 3.5 tells us that the existence of a dominating form implies trace norm convergence of the semigroup. This is remarkable because, even though form domination implies domination for the resolvents, it does in general not imply domination for the semigroups. In fact, for $A := \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $B := \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$ we have $0 \leq A \leq B$, but $e^{-B} \not\leq e^{-A}$ in the sense of positive definiteness. The authors are grateful to Ulrich Groh (Tübingen) for pointing out this example.

Finally, we also consider convergence of semigroups in Schatten norms as operators between different Hilbert spaces. We obtain our main result as a consequence of an interpolation theorem for Schatten class operators. A criterion which enables us to check the trace norm convergence required in the following theorem was given in Theorem 3.5.

Theorem 3.7. *Let $p \in [1, \infty)$. Let $(A_n), A$ be uniformly m -sectorial operators on H , i.e., m -sectorial operators with uniform constants, which generate Gibbs semigroups. Assume that there exists a subspace \tilde{H} of H such that*

- \tilde{H} is a Hilbert space,
- \tilde{H} is compactly embedded in H , and
- there exists some $k \in \mathbb{N}$ such that $D(A_n^k) \subset \tilde{H}$ for all $n \in \mathbb{N}$ with uniform embedding constants.

If

$$\lim_{n \rightarrow \infty} e^{-tA_n} = e^{-tA} \quad \text{in } \mathcal{L}_p(H)$$

for every $t > 0$, then

$$\lim_{n \rightarrow \infty} e^{-tA_n} = e^{-tA} \quad \text{in } \mathcal{L}_q(H, H_\theta)$$

for every $t > 0$ and every $\theta \in (0, 1)$, where $q \in [1, \infty)$ is given by

$$\frac{1}{q} = \frac{\theta}{p} + (1 - \theta)$$

and where H_θ denotes the complex interpolation space $[\tilde{H}, H]_\theta$.

Proof. We first show that $D(A^k)$ is continuously embedded into \tilde{H} . Fix $\lambda > 0$ so large that $\lambda + A_n$ is invertible with uniformly bounded inverse with respect to $n \in \mathbb{N}$. Take $u \in H$. Then the uniform constants in the m -sectoriality and the embeddings ensure that the sequence $((\lambda + A_n)^{-k}u)_{n \in \mathbb{N}}$ is bounded in \tilde{H} . Hence there exists a weakly convergent subsequence in \tilde{H} , which necessarily converges to $(\lambda + A)^{-k}u$ since the semigroups and hence the resolvents converge strongly by assumption. This proves that $D(A^k) \subset \tilde{H}$. Now the closed graph theorem yields $D(A^k) \hookrightarrow \tilde{H}$.

For every $t > 0$ and every $n \in \mathbb{N}$ the operator $e^{-tA_n} = e^{-\frac{t}{2}A_n}e^{-\frac{t}{2}A_n}$ is a composition of an operator in $\mathcal{L}_1(H)$ and an operator in $\mathcal{L}(H, \tilde{H})$, both with uniformly estimable norms, compare (1) in Remarks 3.4. Hence $\sup_{n \in \mathbb{N}} \|e^{-tA_n}\|_{\mathcal{L}_1(H, \tilde{H})} < \infty$ by the ideal property of the norm, and by a similar argument e^{-tA} is in $\mathcal{L}_1(H, \tilde{H})$ as well.

Now we obtain from an interpolation result for Schatten class operators [22] that

$$\|e^{-tA_n} - e^{-tA}\|_{\mathcal{L}_q(H, H_\theta)} \leq C \|e^{-tA_n} - e^{-tA}\|_{\mathcal{L}_1(H, \tilde{H})}^\theta \|e^{-tA_n} - e^{-tA}\|_{\mathcal{L}_p(H)}^{1-\theta},$$

for some constant $C \geq 1$ since the fractional domain space considered in [22] coincides with H_θ up to equivalent norms. The first factor is bounded by the above considerations whereas the second factor converges to zero by assumption. \square

Let us combine several of our observations into a final result.

Corollary 3.8. *Let X be a finite measure space. Let (a_n, j_n) , (a, j) and (b, j') be positive elliptic forms on $L^2(X)$ in the sense of Definition 2.1 with form domain V , and denote the associated self-adjoint operators by A_n , A and B , respectively. Let \tilde{H} be a dense subspace of H , which is a Hilbert space in its own right. Assume that*

- (a_n, j_n) converges to (a, j) in the sense of Mosco;
- $(b, j') \leq (a_n, j_n)$ for all $n \in \mathbb{N}$;
- there exists $q > 2$ such that $j'(V) \subset L^q(X)$;
- for all $u \in V$ there exists $w \in V$ such that $(|j(u)| \wedge 1) \operatorname{sign} j(u) = j(w)$ and $\operatorname{Re} b(w, u - w) \geq 0$;
- \tilde{H} is compactly embedded into H ;
- $D(A_n^k) \subset \tilde{H}$ for some $k \in \mathbb{N}$ with an embedding constant that is uniform in $n \in \mathbb{N}$;

For arbitrary $\theta \in (0, 1)$ let H_θ denote the complex interpolation space $H_\theta = [\tilde{H}, H]_\theta$. Then $e^{-tA_n} \rightarrow e^{-tA}$ in the trace norm $\mathcal{L}_1(H, H_\theta)$ and hence in particular in the operator norm $\mathcal{L}(H, H_\theta)$.

Proof. By the invariance criterion for j -elliptic forms [3, Prop 2.9] the semigroup is $L^\infty(X)$ -contractive, analogously to the situation in [36, Thm. 2.13]. Hence by Proposition 2.12 it is ultra-contractive and thus Gibbs by Remark 3.4. Since in addition $(-A_n)$ converges to $-A$ in the strong resolvent sense by Theorem 3.1 we obtain from Theorem 3.5 that $e^{-tA_n} \rightarrow e^{-tA}$ in $\mathcal{L}_1(H)$. Now the assertion follows from Theorem 3.7. \square

Remark 3.9. The assumption of Corollary 3.8 that $D(A_n^k) \subset \tilde{H}$ with uniform embeddings is in particular satisfied for $\tilde{H} = V$ if the constants in the ellipticity estimate (2.1) of (a_n, j_n) are uniform in n , for the semigroups (e^{tA_n}) are bounded as operators from H to V , uniformly in n .

We emphasise that in this special case Corollary 3.8 yields a convergence result for semigroups under assumptions solely on the associated forms, with no reference to the associated operators.

Remarks 3.10. Let us finally remark on the Gibbs property for other kinds of operator families.

- (1) Let $-A$ be a self-adjoint operator, hence the generator of a sine operator function $(S(t))_{t \in \mathbb{R}}$, cf. [6, § 3.15]. It is known that $S(t)$ maps H into V for all $t \in \mathbb{R}$, where V is the domain of the form associated with A . If the embedding of V into H is of p -Schatten class (e.g., V a closed subspace of $H^1(0, 1)$, $H = L^2(0, 1)$ and $p > 1$, cf. Remark 3.4.(5)), then $S(t)$ is of p -Schatten class for all $t \in \mathbb{R}$.
- (2) Unlike in the semigroup case, however, there exist sine operator functions $(S(t))_{t \in \mathbb{R}}$ on a Hilbert space H such that $S(t) \in \mathcal{L}_p(H)$ for all $t \in \mathbb{R}$ for some $p > 1$, but $S(t) \notin \mathcal{L}_{p-\varepsilon}(H)$ for all $t \in \mathbb{R}$ and all $\varepsilon > 0$. In fact, fix $\alpha \geq 1$ and consider the multiplication operator M_λ on ℓ^2 , where the sequence λ is given by

$$\lambda_n := -\left(\frac{\pi}{2} + 2\pi \lfloor n^\alpha \rfloor\right)^2, \quad n \in \mathbb{N}$$

and $\lfloor x \rfloor$ denotes the greatest integer below x . Then the corresponding sine operator function is given by

$$S(t)x := \left(\frac{\sinh(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}x_n\right)_{n \in \mathbb{N}} = \left(\frac{\sin((\frac{\pi}{2} + 2\pi \lfloor n^\alpha \rfloor)t)}{\frac{\pi}{2} + 2\pi \lfloor n^\alpha \rfloor}x_n\right)_{n \in \mathbb{N}}, \quad t \in \mathbb{R}, x \in \ell^2,$$

so $S(t) \in \mathcal{L}_p(\ell^2)$ for all $p > \alpha^{-1}$ and all $t \in \mathbb{R}$, but $S(1)$ is not in $\mathcal{L}_{\alpha^{-1}}(\ell^2)$.

- (3) On an infinite dimensional Hilbert space a cosine operator function with unbounded generator never consists of Schatten class operators on a non-void open interval. In fact, a cosine operator function can only be compact on an interval of positive length if its generator is a bounded operator, cf. [42, Lemma 2.1].

4. APPLICATIONS

4.1. Convergence of Laplacians with respect to higher regularity Schatten norms. We begin with an application of our result about Schatten convergence, which shows how our convergence results can be combined to treat semigroups generated by elliptic operators: starting with convergence in the strong resolvent sense we are able to obtain trace norm convergence with respect to Sobolev spaces of arbitrarily high order.

Theorem 4.1. *Let Ω be a bounded open domain in \mathbb{R}^d with C^∞ -boundary. Consider a sequence of Laplacians Δ_{k_n} with Robin boundary conditions*

$$\frac{\partial u}{\partial \nu} + k_n u = 0 \quad \text{on } \partial\Omega.$$

for constants $(k_n)_{n \in \mathbb{N}} \subset [0, \infty)$. If (k_n) is a monotonically decreasing null sequence, then

$$\lim_{n \rightarrow \infty} e^{t\Delta_{k_n}} = e^{t\Delta_N} \quad \text{in } \mathcal{L}_1(L^2(\Omega), H^\ell(\Omega))$$

for every $t > 0$ and every $\ell \in \mathbb{N}$, where Δ_N denotes the Laplace operator on Ω with Neumann boundary conditions.

Proof. By [27, Thm. 8.3.11] the sequence (Δ_{k_n}) converges to Δ_N in the strong resolvent sense. Let (a_n) and a_N be the elliptic classical forms associated with $-\Delta_{k_n}$ and $-\Delta_N$, respectively. Then $a_n \geq a_N$ in the sense of Proposition 2.9. Moreover, Δ_N generates a Gibbs semigroup, see (2) in Remarks 3.4 and use [36, Corollary 2.17 and Theorem 6.4]. Hence $e^{-t\Delta_{k_n}} \rightarrow e^{-t\Delta_N}$ in $\mathcal{L}_1(L^2(\Omega))$ by Theorem 3.5. Moreover, following the proofs of elliptic regularity, cf. [26, §2.5.1], one can see that $D(\Delta_{k_n}^\ell)$ is uniformly embedded into $H^{2\ell}(\Omega)$ for every $\ell \in \mathbb{N}$. Applying Theorem 3.7 with $\theta = \frac{1}{2}$ we conclude that $e^{t\Delta_{k_n}} \rightarrow e^{t\Delta_N}$ in $\mathcal{L}_1(L^2(\Omega), H^\ell(\Omega))$ for every $t > 0$ and every $\ell \in \mathbb{N}$. \square

Remark 4.2. Analogous arguments work for heat equations with the dynamic boundary conditions

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial \nu} - k_n u \quad \text{on } \partial\Omega,$$

which arise from forms as seen in [7]. This complements the results of [16], where the emphasis lies in obtaining sharp estimates for the rate of convergence with respect to the H^1 -operator norm.

4.2. Convergence of Laplacians with variable boundary conditions on exterior domains. The result in this section is somewhat special, since we prove Schatten norm convergence of diffusion semigroups $(T_n(t))_{t \geq 0}$ to a semigroup $(T(t))_{t \geq 0}$, all acting on spaces of functions on exterior domains with varying boundary conditions. As we will see, in this situation it is sometimes possible to obtain that $T_n(t) - T(t) \rightarrow 0$ in \mathcal{L}_p (for sufficiently large values of p) as $n \rightarrow \infty$ even though the operators $T_n(t)$ and $T(t)$ are not individually in \mathcal{L}_p and in fact not even compact. In particular, Theorem 3.5 does not apply here. Instead, our argument relies upon classical results on differences of differential operators first due to Mikhail Š. Birman [12, Thm. 3.8] and recently improved in [10]; only Theorem 3.1 is additionally needed. Such situations indeed appear frequently in mathematical physics, see for example [14, 18, 41].

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be an exterior domain with smooth boundary and Δ_β the Laplace operator on Ω with Robin boundary condition*

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega.$$

If $(\beta_n)_{n \in \mathbb{N}}$ is a bounded sequence in $L^\infty(\partial\Omega)$ and converges to a function β_0 almost everywhere, then

$$\lim_{n \rightarrow \infty} (e^{t\Delta_{\beta_n}} - e^{t\Delta_{\beta_0}}) = 0 \quad \text{in } \mathcal{L}_p(L^2(\Omega))$$

for every $t > 0$ and all $p > \frac{d-1}{3}$.

Proof. Let us first show that the operators are uniformly m -sectorial. Since the trace operator $u \mapsto u|_{\partial\Omega}$ is compact from $H^1(\Omega)$ to $L^2(\partial\Omega)$, by Lemma 2.7 there exists $c > 0$ such that

$$\|u\|_{L^2(\partial\Omega)}^2 \leq \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 + c \|u\|_{L^2(\Omega)}^2$$

for all $u \in H^1(\Omega)$. This shows that the quadratic form q_β associated with $-\Delta_\beta$, i.e.

$$q_\beta(u) := \int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} \beta_n |u|^2$$

for $u \in H^1(\Omega)$, is semi-bounded for every essentially bounded function β and hence that Δ_β generates a C_0 -semigroup on $L^2(\Omega)$. More precisely,

$$q_{\beta_n}(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 - cm \|u\|_{L^2(\Omega)}^2$$

with $m := \sup_{n \in \mathbb{N}} \|\beta_n\|_\infty$. This proves uniform m -sectoriality.

Next we show convergence in the strong resolvent sense. We write $\tilde{q}_\beta(u) := q_\beta(u) + (cm + 1) \|u\|_{L^2(\Omega)}^2$ for simplicity and show that \tilde{q}_{β_n} converges to $\tilde{q}_{\beta_0} + cm + 1$ in the sense of Mosco. To this end, let (u_n) be a sequence in $H^1(\Omega)$ such that $u_n \rightharpoonup u$ in $L^2(\Omega)$ and $\liminf_{n \rightarrow \infty} \tilde{q}_{\beta_n}(u_n) < \infty$. Then (u_n) is bounded in $H^1(\Omega)$, hence passing to a subsequence we can assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. Thus

in particular $u \in H^1(\Omega)$. Now by compactness $u_n|_{\partial\Omega} \rightarrow u|_{\partial\Omega}$ in $L^2(\partial\Omega)$, so in particular

$$\int_{\partial\Omega} \beta_n |u_n|^2 \rightarrow \int_{\partial\Omega} \beta_0 |u|^2.$$

By weak lower semicontinuity of the norm of $H^1(\Omega)$ this proves

$$\tilde{q}_{\beta_0}(u) \leq \liminf_{n \rightarrow \infty} \tilde{q}_{\beta_n}(u_n).$$

Moreover, for given $u \in H^1(\Omega)$ we clearly have $\tilde{q}_{\beta_n}(u) \rightarrow \tilde{q}_{\beta_0}(u)$ by Lebesgue's theorem. We thus have shown that $\Delta_{\beta_n} \rightarrow \Delta_{\beta_0}$ in the strong resolvent sense, see Theorem 3.1.

We now prove the convergence in Schatten norm. Since $-\Delta_{-m} \leq -\Delta_{\beta_n} \leq -\Delta_m$ in the form sense we have

$$(cm + 1 - \Delta_m)^{-1} \geq (cm + 1 - \Delta_{\beta_n})^{-1} \geq (cm + 1 - \Delta_{-m})^{-1}$$

as self-adjoint operators [27, Thm. 2.21]. A similar assertion holds for Δ_{β_0} . Consequently,

$$|(cm + 1 - \Delta_{\beta_n})^{-1} - (cm + 1 - \Delta_{\beta_0})^{-1}| \leq (cm + 1 - \Delta_m)^{-1} - (cm + 1 - \Delta_{-m})^{-1},$$

where the right hand side is in $\mathcal{L}_p(L^2(\Omega))$ for every $p > \frac{d-1}{3}$ by [10, Cor. 3.6]. This implies that $(cm + 1 - \Delta_{\beta_n})^{-1}$ converges to $(cm + 1 - \Delta_{\beta_0})^{-1}$ in $\mathcal{L}_p(L^2(\Omega))$ as $n \rightarrow \infty$, see [35, Prop. 2.1].

Moreover, as in the proof of [10, Thm. 3.5], for all λ and μ in the sector $\Sigma := \mathbb{C} \setminus \mathbb{R}_{\leq cm}$ we have

$$\begin{aligned} & (\mu - \Delta_{\beta_n})^{-1} - (\mu - \Delta_{\beta_0})^{-1} \\ &= (I + (\lambda - \mu)(\mu - \Delta_{\beta_0})^{-1})((\lambda - \Delta_{\beta_n})^{-1} - (\lambda - \Delta_{\beta_0})^{-1})(I + (\lambda - \mu)(\mu - \Delta_{\beta_n})^{-1}). \end{aligned}$$

In fact, this identity is certainly satisfied on $V = H^1(\Omega)$ since $\kappa - \Delta_{\beta_n}$ and $\kappa - \Delta_{\beta_0}$ are isomorphisms from V to the dual space V' . Thus the identity extends to $L^2(\Omega)$ by denseness.

Picking $\lambda = cm + 1$ we obtain from the ideal property that

$$(\mu - \Delta_{\beta_n})^{-1} - (\mu - \Delta_{\beta_0})^{-1} \in \mathcal{L}_p(L^2(\Omega))$$

for all $\mu \in \Sigma$. More precisely we even obtain that on every sector smaller than Σ this sequence of differences is bounded and convergent in $\mathcal{L}_p(L^2(\Omega))$ on compact subsets of Σ , uniformly with respect to n . Here we have used that the operators Δ_{β_n} are uniformly m -sectorial. Hence the integral representation [6, (3.46)]

$$e^{t\Delta_{\beta_n}} - e^{t\Delta_{\beta_0}} = \frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} ((\lambda - \Delta_{\beta_n})^{-1} - (\lambda - \Delta_{\beta_0})^{-1}) d\lambda,$$

shows that $e^{t\Delta_{\beta_n}}$ converges to $e^{t\Delta_{\beta_0}}$ in $\mathcal{L}_p(L^2(\Omega))$ for every $t > 0$. \square

4.3. Coupled boundary conditions. In this subsection we consider convergence for systems of Laplacians with a certain coupled boundary conditions, which are motivated by quantum graphs. It seems that [27, Thm. VI.3.6] cannot be used to obtain strong convergence of the resolvents in this example, so we employ an approach developed by Olaf Post [37] instead, where we use the notation of [34].

Theorem 4.4. *Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence of closed subspaces of \mathbb{C}^k , $k \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be an exterior domain with smooth compact boundary and $(\Delta_{Y_n})_{n \in \mathbb{N}}$ be a sequence of Laplacians on $L^2(\Omega; \mathbb{C}^k)$ with boundary conditions*

$$u|_{\partial\Omega} \in Y_n \quad \text{and} \quad \frac{\partial u}{\partial \nu} \in Y_n^\perp \quad \text{a.e., } n \in \mathbb{N}.$$

Assume that there exist a subspace Y of \mathbb{C}^k and a family $(J^{\downarrow n})_{n \in \mathbb{N}}$ of unitary operators on H converging to the identity I such that $J^{\downarrow n} Y_n = Y$ for all $n \in \mathbb{N}$. Denote by Δ_Y the Laplacian with corresponding boundary conditions. Then

$$\lim_{n \rightarrow \infty} e^{t\Delta_{Y_n}} = e^{t\Delta_Y} \quad \text{in } \mathcal{L}_p(L^2(\Omega; \mathbb{C}^k))$$

for every $t > 0$ and all $p > \frac{d-1}{2}$.

Proof. We introduce elliptic forms $(a_n)_{n \in \mathbb{N}}$ and a_0 with form domains

$$\begin{aligned} V_n &:= \{f \in H^1(\Omega; \mathbb{C}^k) : f|_{\partial\Omega} \in Y_n\} \quad n \in \mathbb{N}, \\ V &:= \{f \in H^1(\Omega; \mathbb{C}^k) : f|_{\partial\Omega} \in Y\} \end{aligned}$$

as in [13, §2]. These forms are symmetric, and accordingly the associated Laplacians Δ_{Y_n} and Δ_Y are self-adjoint operators on $H := L^2(\Omega; \mathbb{C}^k)$.

We set

$$\mathcal{J}^{\downarrow n} f := \mathcal{J}_1^{\downarrow n} f := J^{\downarrow n} \circ f \quad \text{and} \quad \mathcal{J}^{\uparrow n} f := \mathcal{J}_1^{\uparrow n} f := J^{\uparrow n} \circ f, \quad n \in \mathbb{N}.$$

Since these operators are unitary on $H = L^2(\Omega; \mathbb{C}^k)$ as well as from V_n to V , it is easy to see that the assumptions in [34, Def. 2.3] are satisfied, and we deduce from [34, Prop. 3.4] that Δ_{Y_n} converges to Δ_Y in the norm resolvent sense.

Moreover, $-\Delta_{\mathbb{C}^k} \leq -\Delta_{Y_n} \leq -\Delta_{\{0\}}$ in the form sense and hence

$$|(\lambda - \Delta_{Y_n})^{-1} - (\lambda - \Delta_{Y_0})^{-1}| \leq (\lambda - \Delta_{\{0\}})^{-1} - (\lambda - \Delta_{\mathbb{C}^k})^{-1}$$

by [27, Thm. 2.21]. Using that $\Delta_{\{0\}}$ and $\Delta_{\mathbb{C}^k}$ act as uncoupled copies of k Dirichlet and Neumann Laplace operators, respectively, we obtain from Birman's result [12, Thm. 3.8] that the operator on the right hand side is in $\mathcal{L}_p(L^2(\Omega))$ for every $p > \frac{d-1}{2}$. The conclusion now follows as in Theorem 4.3. \square

Remark 4.5. We have formulated Theorems 4.3 and 4.4 in the case of Laplacians only for the sake of simplicity: in fact, both results can be extended to strongly elliptic operators with coefficients in $W^{1,\infty}(\Omega)$. Also, rougher and even non-compact boundaries can be allowed, leading to convergence only for larger p , cf. [12, Rem. 3.4, Thm. 3.8 and Thm. 5.2].

4.4. Dirichlet-to-Neumann-type operators. By showing that the Dirichlet-to-Neumann operator is associated with a j -elliptic form, Arendt and ter Elst have delivered a most interesting application of their theory. This is an instance where a non-injective j appears in a natural way.

Consider an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, where $d \geq 2$, and let $V := H^1(\Omega)$ and $H := L^2(\partial\Omega)$. We consider the sesquilinear form a defined by

$$a(u, v) := \int_{\Omega} \alpha \nabla u \cdot \overline{\nabla v}, \quad u, v \in V,$$

where the matrix-valued coefficient $\alpha \in L^\infty(\Omega; \mathbb{C}^{d \times d})$ is uniformly positive definite, i.e., for a.e. $x \in \Omega$ the matrix $\alpha(x)$ is Hermitian and satisfies

$$(\alpha(x)\xi|\xi) \geq k_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^d$$

for some $k_0 > 0$. Let j be the trace operator from V to H . It can be checked as in [3, §4.4] that a is a j -elliptic symmetric form, and more precisely

$$a(u, u) - \omega \|j(u)\|_H^2 \geq \mu \|u\|_V^2 \quad \text{for all } u \in V$$

for some $\omega \in \mathbb{R}$ and $\mu > 0$.

Let $\lambda_1^D(a)$ denote the smallest eigenvalue of the operator associated with the restriction of a to $H_0^1(\Omega)$, i.e.,

$$\lambda_1^D(a) := \inf_{u \in H_0^1(\Omega)} \frac{a(u, u)}{\|u\|_{L^2(\Omega)}^2}.$$

Let $\gamma_0 < \lambda_1^D(a)$ and fix a complex-valued function $\gamma \in L^q(\Omega)$, $q > \frac{d}{2}$, satisfying $\operatorname{Re} \gamma \geq -\gamma_0$. Then

$$b(u, v) := \int_{\Omega} \gamma u \bar{v}, \quad u, v \in V,$$

defines a bounded sesquilinear form on V by the Hölder inequality and the Sobolev embedding theorem.

Proposition 4.6. *Under the above assumptions, $a + b$ is j -elliptic.*

Proof. By the variational characterisation of λ_1^D we have for all $u \in H_0^1(\Omega)$ that

$$\operatorname{Re} b(u, u) \geq -\gamma_0 \int_{\Omega} |u|^2 \geq -\frac{\gamma_0}{\lambda_1^D(a)} a(u, u)$$

and hence

$$(4.1) \quad a(u, u) + \operatorname{Re} b(u, u) \geq \tilde{\eta} a(u, u) \geq \tilde{\eta} k_0 \int_{\Omega} |\nabla u|^2 \geq \eta \|u\|_V^2$$

for all $u \in H_0^1(\Omega)$, where $\tilde{\eta} := 1 - \frac{\gamma_0}{\lambda_1^D(a)} > 0$ and $\eta > 0$ depends on the first eigenvalue of the Dirichlet Laplacian on Ω . Moreover, $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ and j is injective on $V(a)$, hence for all $u \in V(a)$ we have

$$\operatorname{Re} b(u, u) \geq -\gamma_0 \int_{\Omega} |u|^2 \geq \frac{\mu}{2} \|u\|_V^2 - c_{\mu} \|j(u)\|_H^2$$

for some $c_{\mu} \geq 0$ by Lemma 2.7 and hence

$$(4.2) \quad a(u, u) + \operatorname{Re} b(u, u) - (\omega - c_{\mu}) \|j(u)\|_H^2 \geq \frac{\mu}{2} \|u\|_V^2$$

for all $u \in V(a)$.

For $u \in H_0^1(\Omega)$ and $v \in V(a)$ we have $a(v, u) = a(u, v) = 0$ by definition of $V(a)$. Moreover, for every $\varepsilon > 0$ we have

$$(4.3) \quad \begin{aligned} |b(u, v)| + |b(v, u)| &\leq c \|u\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)} \leq \frac{c\varepsilon}{2} \|u\|_{L^p(\Omega)}^2 + \frac{c}{2\varepsilon} \|v\|_{L^p(\Omega)}^2 \\ &\leq \frac{c\varepsilon}{2} \|u\|_V^2 + c_{\varepsilon} \|j(v)\|_H^2 + \varepsilon \|v\|_V^2 \end{aligned}$$

for some $p \in [2, \frac{2(d-1)}{d-2})$ and some $c, c_{\varepsilon} \geq 0$ by the integrability assumptions on γ , the Sobolev embeddings theorems and Lemma 2.7.

Since every $u \in V$ has a representation of the form $u = u_1 + u_2$ with $u_1 \in H_0^1(\Omega)$ and $u_2 \in V(a)$ by Remark 2.2, combining (4.1), (4.2) and (4.3), where in the latter we pick $\varepsilon > 0$ such that $\frac{c\varepsilon}{2} < \frac{\eta}{2}$ and $\varepsilon < \frac{\mu}{4}$, we obtain that

$$\begin{aligned} a(u, u) + \operatorname{Re} b(u, u) &= a(u_1, u_1) + \operatorname{Re} b(u_1, u_1) + a(u_2, u_2) + \operatorname{Re} b(u_2, u_2) + \operatorname{Re} b(u_1, u_2) + \operatorname{Re} b(u_2, u_1) \\ &\geq \eta \|u_1\|_V^2 + \frac{\mu}{2} \|u_2\|_V^2 + (\omega - c_{\mu}) \|j(u_2)\|_H^2 - \frac{\eta}{2} \|u_1\|_V^2 - \frac{\mu}{4} \|u_2\|_V^2 - c_{\varepsilon} \|j(u_2)\|_H^2 \\ &= \frac{\eta}{2} \|u_1\|_V^2 + \frac{\mu}{4} \|u_2\|_V^2 + \omega' \|j(u_2)\|_H^2 \end{aligned}$$

for some $\omega' \in \mathbb{R}$. Finally, since $V = H_0^1(\Omega) \oplus V(a)$ by Remark 2.2, the expression $\|u\|^2 := \|u_1\|_V^2 + \|u_2\|_V^2$ defines an equivalent norm on V , which allows us to write the previous estimate as

$$a(u, u) + \operatorname{Re} b(u, u) - \omega' \|j(u)\|_H^2 \geq \mu' \|u\|_V^2 \quad \text{for all } u \in V,$$

for some $\mu' > 0$. This is the j -ellipticity of $a + b$. \square

Following [3, §4.4] one can check that the operator $-D_\alpha^\gamma$ associated with $a + b$ is some *Dirichlet-to-Neumann* operator. More precisely, $\varphi \in L^2(\partial\Omega)$ is in $D(D_\alpha^\gamma)$ if and only if there exists a (necessarily unique) weak solution of the inhomogeneous Dirichlet problem

$$(IDP) \quad \begin{cases} \gamma u - \operatorname{div}(\alpha \nabla u) = 0, & x \in \Omega, \\ u(z) = \varphi(z), & z \in \partial\Omega, \end{cases}$$

and the weak conormal derivative $\frac{\partial u}{\partial \nu_\alpha}$ exists as an element of $L^2(\partial\Omega)$. In this case, $-D_\alpha^\gamma u = \frac{\partial u}{\partial \nu_\alpha}$. The above considerations show that D_α^γ generates an analytic semigroup. We formulate this as a theorem.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary, where $d \geq 2$. Let $\alpha \in L^\infty(\Omega; \mathbb{C}^{d \times d})$ be uniformly positive definite, and let $\gamma \in L^q(\Omega; \mathbb{C})$, $q > \frac{d}{2}$, be such that $\operatorname{Re} \gamma \geq -\gamma_0$ for some $\gamma_0 < \lambda_1^D$. Then the operator D_α^γ generates an analytic semigroup on $H = L^2(\partial\Omega)$, which is Gibbs if additionally $\gamma \geq 0$.*

If Ω , α and γ are smooth, then the analyticity angle of this semigroup is $\frac{\pi}{2}$.

Proof. Let us prove the assertion on the analyticity angle. Let everything be smooth, so that in particular γ is bounded. By Proposition 2.4 it suffices to check that

$$M \|u\|_{H^1(\Omega)} \|u\|_{L^2(\partial\Omega)} \geq \left| \operatorname{Im} \int_\Omega \gamma |u|^2 \right| \quad \text{for all } u \in V(a + b)$$

holds for some $M \geq 0$, where $V(a + b)$ consists by definition of all H^1 -functions that are weak solutions of (IDP) for some γ . Since $\operatorname{Re} \gamma \geq -\gamma_0$ the only function $u \in H_0^1(\Omega)$ satisfying $\gamma u - \operatorname{div}(\alpha \nabla u) = 0$ is $u = 0$. Hence by [29, Thm. 2.7.4] the trace operator is an isomorphism from

$$\{u \in H^{\frac{1}{2}}(\Omega) : \gamma u - \operatorname{div}(\alpha \nabla u) = 0\}$$

onto $L^2(\partial\Omega)$. Accordingly, the estimate in Proposition 2.4 can be equivalently formulated as

$$M \|u\|_{H^1(\Omega)} \|u\|_{H^{\frac{1}{2}}(\Omega)} \geq \left| \operatorname{Im} \int_\Omega \gamma |u|^2 \right| \quad \text{for all } u \in V(a + b)$$

for some possibly larger constant M . This is satisfied whenever γ is bounded.

Assume now that $\gamma \geq 0$. Then by [3, Prop. 2.9] the Dirichlet-to-Neumann semigroup of Theorem 4.7 submarkovian, i.e., positive and $L^\infty(\partial\Omega)$ -contractive, which is easily checked by a version of an invariance criterion due to Ouhabaz for j -elliptic forms [3, Prop. 2.9], see also [4, Prop. 3.7]. In this case Proposition 2.12 and the Sobolev embedding theorems for $\partial\Omega$ (see e.g. [9, Thm. 2.20]) yield in particular that the Dirichlet-to-Neumann semigroup is a Gibbs semigroup. \square

Remark 4.8. For the last step of the preceding proof we only need that $\gamma \in L^{\frac{2d}{3}}(\Omega)$. Hence one could suspect that for all such γ the operator D_α^γ generates a cosine operator function without any additional conditions on the smoothness of α , Ω and γ . However, to extend the result to this situation we would need a generalisation of [29, Thm. 2.7.4] to rough domains and rough coefficients. A partial result into this direction is [23, Lemma. 3.1], where for the Laplace operator [29, Thm. 2.7.4] is extended to Lipschitz domains.

Remark 4.9. We regard the perturbation we are considering as interesting mainly because it cannot be expressed as a perturbation by an operator. In comparison, if for smooth Ω we consider the vaguely related sesquilinear form $b' : V \times V \rightarrow \mathbb{C}$

defined by $b'(u, v) := \int_{\partial\Omega} \beta u \bar{v}$ with $\beta \in L^{d-1}(\partial\Omega)$, then $a + b'$ is associated with $-D_\alpha^\gamma - B$, where B is a bounded operator from $D(D_\alpha^\gamma)$ to $L^2(\Omega)$, and we can deal with it using perturbation theorems for generators.

Remark 4.10. The Gibbs property of the semigroup in Theorem 4.7 has been observed before by Zagrebnov [45, Lemma 2.14]. His sketch of the proof is based on a Weyl-type asymptotic result for the Dirichlet-to-Neumann operator [45, Prop 2.5], which seems to require smoothness of the boundary. A complete proof is announced for a forthcoming (but not yet accessible) joint paper with Hassan Emamirad.

In the self-adjoint case we can also prove the following convergence result.

Theorem 4.11. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary, where $d \geq 2$. Let $(\alpha_n)_{n \in \mathbb{N}} \subset L^\infty(\Omega; \mathbb{C}^{d \times d})$ be such that $\alpha_n(x)$ is uniformly positive definite uniformly with respect to n , i.e.,*

$$(\alpha_n(x)\xi|\xi) \geq k_0|\xi|^2 \quad \text{for a.e. } x \in \Omega, \text{ all } n \in \mathbb{N} \text{ and all } \xi \in \mathbb{C}^d$$

for some $k_0 > 0$. Let finally $(\gamma_n)_{n \in \mathbb{N}} \subset L^q(\Omega; \mathbb{C})$, $q > \frac{d}{2}$, be such that $\operatorname{Re} \gamma_n \geq -\gamma_0$ a.e. for some

$$\gamma_0 < \inf_{n \in \mathbb{N}} \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega \alpha_n \nabla u \cdot \overline{\nabla u}}{\|u\|_{L^2(\Omega)}^2}.$$

If $\lim_{n \rightarrow \infty} \gamma_n = \gamma$ and $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ almost everywhere, then

$$\lim_{n \rightarrow \infty} e^{-tD_{\alpha_n}^{\gamma_n}} = e^{-tD_\alpha^\gamma} \quad \text{in } \mathcal{L}_1(L^2(\partial\Omega))$$

for every $t > 0$.

Proof. Define

$$b(u, v) := k_0 \int_\Omega \nabla u \cdot \overline{\nabla v} - \gamma_0 \int_\Omega u \bar{v}$$

for $u, v \in H^1(\Omega)$, and let j be the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. Then $(b, j) \leq (a_n, j)$ in the sense of Proposition 2.9 for all $n \in \mathbb{N}$, where a_n denotes the form associated with $-D_{\alpha_n}^{\gamma_n}$. Moreover, the semigroup associated with (b, j) is a Gibbs semigroup by Remark 4.10. So in view of Theorem 3.5 it only remains to show that $D_{\alpha_n}^{\gamma_n} \rightarrow D_\alpha^\gamma$ in the strong resolvent sense, for which we employ Remark 3.2.

Let (u_n) be a sequence in $H^1(\Omega)$ such that $u_n|_{\partial\Omega} \rightharpoonup \varphi$ in $L^2(\Omega)$ and $s := \liminf a_n(u_n, u_n) < \infty$. Since the constants in the j_n -ellipticity of a_n are uniform with respect to n , the sequence (u_n) is bounded in $V = H^1(\Omega)$, and thus we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$ for some $u \in H^1(\Omega)$. Then by compactness $\lim_{n \rightarrow \infty} u_n = u$ in $L^2(\Omega)$ and moreover $u_n|_{\partial\Omega} \rightarrow u|_{\partial\Omega}$ in $L^2(\partial\Omega)$. From this and weak lower semicontinuity of the norm in $H^1(\Omega)$ it follows immediately that $a(u, u) \leq s$, where a denotes the form associated with $-D_\alpha^\gamma$. Moreover, if $u \in H^1(\Omega)$, then clearly $a_n(u_n, u_n) \rightarrow a(u, u)$. Now the convergence follows from Remark 3.2. \square

4.5. Multiplicative perturbations of Laplacians. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set. Let $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. Define $a(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v}$ and $j(u) := \frac{u}{m}$, where the real-valued function m on Ω satisfies $0 < \varepsilon \leq m \leq M < \infty$ for some constants ε and M . Then for the operator $-A_m$ associated with (a, j) we have $u \in D(A_m)$ with $-A_m u = f$ if and only if $u \in H_0^1(\Omega)$ and $\Delta u = \frac{f}{m}$ distributionally, i.e., at least symbolically, $A_m = -m\Delta$ with Dirichlet boundary conditions.

Theorem 4.12. *Let $(m_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions from Ω to \mathbb{R} such that $0 < \varepsilon \leq m_n \leq M < \infty$ for all $n \in \mathbb{N}$. If this sequence converges a.e. to a measurable function $m : \Omega \rightarrow \mathbb{R}$, then*

$$\lim_{n \rightarrow \infty} e^{tm_n \Delta} = e^{tm \Delta} \quad \text{in } \mathcal{L}_1(L^2(\Omega))$$

for every $t > 0$.

Proof. Comparing with the Gibbs semigroup generated by $-\varepsilon \Delta$, we see as in the previous section that it suffices to prove convergence of $-m_n \Delta$ to $-m \Delta$ in the strong resolvent sense. So take a sequence (u_n) in $H^1(\Omega)$ such that $m_n u_n \rightarrow mu$ in $L^2(\Omega)$ and $s := \liminf_{n \rightarrow \infty} a(u_n, u_n) < \infty$. Then $u_n \rightarrow u$ in $H^1(\Omega)$ after passing to a subsequence, and hence $\lim_{n \rightarrow \infty} u_n = u$ by compact embedding, which shows in particular that $u \in H^1(\Omega)$. The relation $a(u, u) \leq s$ is obvious from weak lower semicontinuity of the norm in $H^1(\Omega)$. On the contrary, if $u \in H^1(\Omega)$, then $m_n u \rightarrow mu$ in $L^2(\Omega)$. Hence we obtain convergence in the strong resolvent sense from Theorem 3.1. \square

Remark 4.13. For every $k \in \mathbb{N}$, the k^{th} eigenvalue $\lambda_k(A_m)$ of A_m is an increasing function of m . More precisely, if $m_1 \leq m_2$ almost everywhere, then $\|\frac{u}{m_1}\|_2 \geq \|\frac{u}{m_2}\|_2$ for all $u \in H_0^1(\Omega)$ and hence $\lambda_k(A_{m_1}) \leq \lambda_k(A_{m_2})$ by Theorem 2.11. By the way, the operators can in general not be compared in the sense of positive definiteness, as they are not self-adjoint on the same reference space, so the expression $A_{m_1} \leq A_{m_2}$ is not defined and we have to resort to the eigenvalues if we wish to compare the operators in some way.

4.6. Comparison of self-adjoint elliptic operators. Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary. Let $V := H^1(\Omega)$ and define

$$a(u, v) := \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\partial\Omega} \beta u \bar{v} d\sigma$$

for a given real-valued function $\beta \in L^\infty(\partial\Omega)$. We consider the operators $j_1 : V \rightarrow L^2(\Omega)$, $j_2 : V \rightarrow L^2(\Omega) \times L^2(\partial\Omega)$ and $j_3 : V \rightarrow L^2(\partial\Omega)$ given by $j_1(u) := u$, $j_2(u) := (u, u|_{\partial\Omega})$ and $j_3(u) := u|_{\partial\Omega}$, respectively. Then a is a j_k -elliptic form and we denote the operator associated with (a, j_k) by A_k , $k = 1, 2, 3$. These operators are given by

$$\begin{aligned} u \in D(A_1), A_1 u = f &\Leftrightarrow \begin{cases} -\Delta u = f \\ \frac{\partial u}{\partial \nu} + \beta u = 0 \end{cases} \\ (u, u|_{\partial\Omega}) \in D(A_2), A_2(u, u|_{\partial\Omega}) = (f, g) &\Leftrightarrow \begin{cases} -\Delta u = f \\ \frac{\partial u}{\partial \nu} + \beta u = g \end{cases} \\ \varphi \in D(A_3), A_3 \varphi = g &\Leftrightarrow \exists u \in H^1(\Omega) : \begin{cases} -\Delta u = 0 \\ u|_{\partial\Omega} = \varphi \\ \frac{\partial u}{\partial \nu} + \beta u = g \end{cases} \end{aligned}$$

where the Laplace operator and the normal derivative are understood in a weak sense, see [3, §4.4] for A_3 .

Now Theorem 2.11 yields that

$$\lambda_k(A_2) \leq \lambda_k(A_1) \quad \text{and} \quad \lambda_k(A_2) \leq \lambda_k(A_3) \quad \text{for all } k \in \mathbb{N}.$$

These results have also been obtained in [11, Thm. 4.2 and Thm. 4.3] by the same argument.

4.7. Convergence and non-convergence of Wentzell–Robin operators. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and define $V := \{(u, u|_{\partial\Omega}) : u \in H^1(\Omega)\}$ and $H := L^2(\Omega) \times L^2(\partial\Omega)$. Then

$$a((u, u|_{\partial\Omega}), (v, v|_{\partial\Omega})) := \int_{\Omega} \nabla u \cdot \overline{\nabla v}$$

defines a bounded sesquilinear form on V . We consider the embeddings

$$j_{\rho, \sigma}((u, u|_{\partial\Omega})) := (\rho u, \sigma u|_{\partial\Omega})$$

of V into H , where $\sigma > 0$ and $\rho > 0$ are constants. Then clearly a is a positive $j_{\rho, \sigma}$ -elliptic form, and the associated operator $A_{\rho, \sigma}$ is (at least on a formal level) given by

$$A_{\rho, \sigma}(u, u|_{\partial\Omega}) = \left(-\frac{1}{\rho}\Delta u, \frac{1}{\sigma}\frac{\partial u}{\partial\nu}\right).$$

Theorem 4.14. *The operator $A_{1, \sigma}$ converges to $-\Delta_D \oplus 0$ in the strong resolvent sense as $\sigma \rightarrow \infty$, where Δ_D denotes the Dirichlet Laplacian on $L^2(\Omega)$.*

Proof. The operator $-\Delta_D \oplus 0$ is associated with the j_D -elliptic form a_D given by $a_D((u, g), (v, h)) := \int_{\Omega} \nabla u \cdot \overline{\nabla v}$, where $j_D: H_0^1(\Omega) \times L^2(\partial\Omega) \rightarrow H$ is given by $j_D((u, g)) := (u, g)$.

Let $\sigma_n \rightarrow \infty$ and let (u_n) be a sequence in V such that

$$j_n(u_n) := (u_n, \sigma_n u_n|_{\partial\Omega}) \rightharpoonup (u, g)$$

in H and $s := \liminf \int_{\Omega} |\nabla u_n|^2 < \infty$. Passing to a subsequence we can assume that $\int_{\Omega} |\nabla u_n|^2 \rightarrow s$. Then (u_n) is bounded in $H^1(\Omega)$, and passing to further subsequence we can assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. Then in particular $u_n|_{\partial\Omega} \rightarrow u|_{\partial\Omega}$ and $\int_{\Omega} |\nabla u|^2 \leq s$. Moreover,

$$u_n|_{\partial\Omega} = \frac{\sigma_n u_n|_{\partial\Omega}}{\sigma_n} \rightarrow 0$$

since $(\sigma_n u_n|_{\partial\Omega})$ is bounded and $\sigma_n \rightarrow \infty$, hence $u|_{\partial\Omega} = 0$. Thus $(u, g) \in H_0^1(\Omega) \times L^2(\partial\Omega)$, $j_D((u, g)) = (u, g)$ and $\liminf \int_{\Omega} |\nabla u_n|^2 \geq \int_{\Omega} |\nabla u|^2$. We have checked the first part of the characterisation in Theorem 3.1.

For the second part, let $(u, g) \in H_0^1(\Omega) \times L^2(\partial\Omega)$ be fixed. Since $\sigma_n \rightarrow \infty$, there exist $v_n \in H^1(\Omega)$ satisfying $v_n|_{\partial\Omega} \rightarrow g$ in $L^2(\partial\Omega)$ and $\frac{v_n}{\sigma_n} \rightarrow 0$ in $H^1(\Omega)$. Define $u_n := u + \frac{v_n}{\sigma_n}$. Then $(u_n, u_n|_{\partial\Omega}) \in V$ and

$$j_n((u_n, u_n|_{\partial\Omega})) = (u_n, v_n|_{\partial\Omega}) \rightarrow (u, g) = f_D((u, g))$$

in H . Moreover, $\int_{\Omega} |\nabla u_n|^2 \rightarrow \int_{\Omega} |\nabla u|^2$ since $u_n \rightarrow u$ in $H^1(\Omega)$. \square

As already emphasised, one advantage of our Mosco-type result is that it *characterises* convergence, meaning that it paves the road to *non-convergence* results as well. Given that the eigenvalue problem associated with the operator $A_{\rho, \sigma}$ is

$$\begin{cases} \lambda \rho u = \Delta u & \text{in } \Omega, \\ \lambda \sigma u|_{\partial\Omega} = -\frac{\partial u}{\partial\nu} & \text{on } \partial\Omega, \end{cases}$$

while the eigenvalue problem associated with the Dirichlet-to-Neumann operator is

$$\begin{cases} 0 = \Delta u & \text{in } \Omega, \\ \lambda u|_{\partial\Omega} = -\frac{\partial u}{\partial\nu} & \text{on } \partial\Omega, \end{cases}$$

the following may look surprising.

Proposition 4.15. *The following assertions hold in the space $L^2(\Omega) \times L^2(\partial\Omega)$.*

- (1) $A_{1,\sigma}$ does not converge to any closed operator in the weak resolvent sense as $\sigma \rightarrow 0$.
- (2) $A_{\rho,1}$ does not converge to any closed operator in the weak resolvent sense as $\rho \rightarrow 0$.

Proof. In both cases, we follow the same strategy. Assume that the family of operators converges to a densely defined (necessarily self-adjoint) operator B on $H := L^2(\Omega) \times L^2(\partial\Omega)$ in the weak resolvent sense. Then the operators converge even in the strong resolvent sense [38, §VIII.7], and hence the quadratic forms converge in the sense of Mosco by Theorem 3.1. But for both situations we will show that the set of u such that the second condition of part (b) of Theorem 3.1 can be satisfied is non-dense in H . Hence there cannot be a limiting quadratic form, thus proving the claim.

(1) Take a null sequence $(\sigma_n)_{n \in \mathbb{N}}$. Let (u_n) be a sequence in $H^1(\Omega)$ such that $j_n((u_n, u_n|_{\partial\Omega})) = (u_n, \sigma_n u_n|_{\partial\Omega})$ converges to (u, g) in H . Assume moreover that $a((u_n, u_n|_{\partial\Omega}), (u_n, u_n|_{\partial\Omega}))$ is bounded. Then $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\Omega)$. Hence $(u_n|_{\partial\Omega})_{n \in \mathbb{N}}$ is bounded in $L^2(\partial\Omega)$, implying that $\sigma_n u_n|_{\partial\Omega} \rightarrow 0$, i.e., $g = 0$. Hence the set of possible limits in (b.ii) of Theorem 3.1 is contained in the non-dense set $L^2(\Omega) \times \{0\}$.

(2) Let $(\rho_n)_{n \in \mathbb{N}}$ be a null sequence. Let (u_n) be a sequence in $H^1(\Omega)$ such that $j_n((u_n, u_n|_{\partial\Omega})) = (\rho_n u_n, u_n|_{\partial\Omega})$ converges to (u, g) in H . Assume moreover that $a((u_n, u_n|_{\partial\Omega}), (u_n, u_n|_{\partial\Omega}))$ is bounded. Then $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\Omega)$ since for some $c > 0$ we have

$$\|u\|_{H^1(\Omega)}^2 \leq c \|\nabla u\|_{L^2(\Omega; \mathbb{R}^N)}^2 + c \|u|_{\partial\Omega}\|_{L^2(\partial\Omega)}^2$$

by [31, §1.1.15]. Hence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, implying that $\rho_n u_n \rightarrow 0$, i.e., $u = 0$. Hence the set of possible limits in (b.ii) of Theorem 3.1 is contained in the non-dense set $\{0\} \times L^2(\partial\Omega)$. \square

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